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Surface Family with a Common Natural Line of Curvature Lift of a Spacelike Curve with Spacelike Binormal in Minkowski 3-Space

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ABSTRACT
We construct a surface family possessing a natural lift of a given spacelike curve with spacelike binormal as a line of curvature. We obtain sufficient condition for the given curve such that its natural lift is a line of curvature on any member of the surface family. Finally, we present an illustrative example.

KEYWORDS
Minkowski 3-space, Line of curvature, Surface family, Natural lift curve.

1. Introduction

We encounter curves and surfaces in every differential geometry book. Regardless of the representation of the surface, most existing work deal with the classification of surface curves. However, the more relevant problem is constructing surfaces upon a given curve possessing it as a special curve rather than finding and classifying surface curves. The first paper related with this type of problem proposed by Wang et al. [17]. They constructed surfaces passing through a given curve as common geodesic. In 2011, a similar paper published by Li et al. [11] who handled the problem of finding surfaces with a common line of curvature. Bayram et al. [4] tackled the problem of constructing surfaces passing through a given asymptotic curve. In [15] authors studied surfaces with a null asymptotic curve. Recently, Bayram et al. [5] found constraints for the natural lift of a given curve to be an asymptotic curve on the surface family. Inspired with the above papers, we search for a surface family possessing the natural lift of a given curve as a common line of curvature. We obtain sufficient condition for the resulting surface to have the natural lift of a given curve as a common line of curvature.

Minkowski 3-space $\mathbb{R}^3_1$ is the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X, X) = -x_1^2 + x_2^2 + x_3^2,$$

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where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ or $X = 0$ and lightlike (or null) if $g(X,X) = 0$ and $x \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{R}^3_1$ can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively timelike, spacelike or null (lightlike), for every $s \in I \subset \mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$). The norm of a vector X is defined by $\|X\|_{IL} = \sqrt{|g(X,X)|}$, [12].

The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ are orthogonal if and only if $g(X,X) = 0$, [14].

Now let X and Y be two vectors in $\mathbb{R}^3_1$, then the Lorentzian cross product is given by [13]

$$\begin{aligned} X \times Y &= \begin{vmatrix} \vec{e_1} & -\vec{e_2} & -\vec{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &= (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2). \end{aligned}$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and the binormal vector fields of the curve α, respectively.

Let α be a unit speed timelike curve with curvature κ and torsion τ. So, T is a timelike vector field, N and B are spacelike vector fields. For these vectors, we can write

$$T \times N = -B, \quad N \times B = T, \quad B \times T = -N,$$

where $\times$ is the Lorentzian cross product in $\mathbb{R}^3_1$ The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}^3_1$. Then, Frenet formulas are given by [16]

$$T' = \kappa N,\ N' = \kappa T + \tau B,\ B' = -\tau N.$$

Let α be a unit speed spacelike curve with spacelike binormal. Now, T and B are spacelike vector fields and N is a timelike vector field. In this situation,

$$T \times N = -B, \quad N \times B = -T, \quad B \times T = N.$$

The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}^3_1$. Then, Frenet formulas are given by [16] .

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Let α be a unit speed spacelike curve with timelike binormal. Now, T and N are spacelike vector fields and B is a timelike vector field. In this situation,

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The binormal vector field $B(s)$ is the unique timelike unit vector field perpendicular to the spacelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}_1^3$. Then, Frenet formulas are given by [6].

$$T' = \kappa N, \; N' = -\kappa T + \tau B, \; B' = \tau N.$$

Lemma 1.0.1. *Let X and Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}_1^3$. If X is timelike, then Y is spacelike [14].*

Lemma 1.0.2. *Let X and Y be positive (negative) timelike vectors in $\mathbb{R}_1^3$. Then*

$$g(X, Y) \leq \|X\| \, \|Y\|$$

whit equality if and only if X and Y are linearly dependent [14].

Lemma 1.0.3. ***i)*** *Let X and Y be positive (negative) timelike vectors in $\mathbb{R}_1^3$. By Lemma 1.0.2, there is a unique nonnegative real number $\varphi(X, Y)$ such that*

$$g(X, Y) = \|X\| \, \|Y\| \cosh \varphi(X, Y).$$

The Lorentzian timelike angle between X and Y is defined to be $\varphi(X, Y)$ [14]. ***ii)*** *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a spacelike vector subspace. Then we have*

$$|g(X, Y)| \leq \|X\| \, \|Y\|.$$

Hence, there is a unique real number $\varphi(X, Y)$ between 0 and π such that

$$g(X, Y) = \|X\| \, \|Y\| \cos \varphi(X, Y).$$

$\varphi(X, Y)$ is defined to be the Lorentzian spacelike angle between X and Y [14].
iii) *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace. Then, we have*

$$g(X, Y) > \|X\| \, \|Y\|.$$

Hence, there is a unique positive real number $\varphi(X, Y)$ between 0 and π such that

$$|g(X, Y)| = \|X\| \, \|Y\| \cosh \varphi(X, Y).$$

$\varphi(X, Y)$ is defined to be the Lorentzian timelike angle between X and Y [14].
iv) *Let X be a spacelike vector and Y be a positive timelike vector in $\mathbb{R}_1^3$. Then there is a unique nonnegative real number $\varphi(X, Y)$ such that*

$$|g(X, Y)| = \|X\| \, \|Y\| \sinh \varphi(X, Y).$$

$\varphi(X, Y)$ is defined to be the Lorentzian timelike angle between X and Y [14].

i) *For the curve γ with a timelike tanget vector field, φ being a Lorentzian timelike angle between the unit spacelike binormal $-B$ and the Frenet instantaneous rotation vector field W,*

a) *If $|\kappa| > |\tau|$, then W is a spacelike vector field. In this situation, from Lemma 3 iii) we can write*

$$\kappa = \|W\| \cosh\theta, \ \tau = \|W\| \sinh\theta$$

$\|W\|^2 = g(W, W) = \kappa^2 - \tau^2$ *and* $C = \frac{W}{\|W\|} = \sinh\varphi T - \cosh\varphi B$, *where C is unit vector field of direction W.*

b) *If $|\kappa| < |\tau|$, then W is a timelike vector field. In this situation, from Lemma 3 iv) we can write*

$$\kappa = \|W\| \sinh\theta, \ \tau = \|W\| \cosh\theta$$

$\|W\|^2 = -g(W, W) = -\left(\kappa^2 - \tau^2\right)$ *and* $C = \cosh\varphi T - \sinh\varphi B$.

ii) *For the curve γ with a timelike principal normal vector field, θ being an angle betwee B and W, if B and W spacelike vector fields that span a spacelike vector subspace then by the Lemma 1.0.3 ii) we can write*

$$\kappa = \|W\| \cos\theta, \quad \tau = \|W\| \sin\theta$$

$\|W\|^2 = g(W, W) = \kappa^2 + \tau^2$ *and* $C = -\sin\varphi T + \cos\varphi B$.

iii) *For the curve γ with a timelike binormal vector field, φ being a Lorentzian timelike angle between $-B$ and W,*

a) *If $|\kappa| < |\tau|$, then W is a spacelike vector field. In this situation, from Lemma 1.0.3 iv) we can write*

$$\kappa = \|W\| \sinh\theta, \ \tau = \|W\| \cosh\theta$$

$\|W\|^2 = g(W, W) = \tau^2 - \kappa^2$ *and* $C = \cosh\varphi T - \sinh\varphi B$.

b) *If $|\kappa| > |\tau|$, then W is a timelike vector field. In this situation, from Lemma 1.0.3 i) we have*

$$\kappa = \|W\| \cosh\theta, \ \tau = \|W\| \sinh\theta$$

$\|W\|^2 = -g(W, W) = -\left(\tau^2 - \kappa^2\right)$ *and* $C = \sinh\varphi T - \cosh\varphi B$.

Let P be a surface in $\mathbb{R}^3_1$ and let $\alpha : I \longrightarrow P$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)), \ (\text{for all } t \in I),$$

where X is a smooth tangent vector field on P [12]. We have

$$TP = \bigcup_{p \in P} T_pP = \chi(P),$$

where T_pP is the tangent space of P at p and $\chi(P)$ is the space of tangent vector fields on P.

For any parametrized curve $\alpha : I \longrightarrow P$, $\bar{\alpha} : I \longrightarrow TP$ is given by

$$\bar{\alpha}(s) = \big(\alpha(s), \alpha'(s)\big) = \alpha'(s)\,|_{\alpha(s)}$$

is called the natural lift of α on TP [6].

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length timelike curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike or spacelike binormal. We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$.

a) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike binormal.

i) If the Darboux vector W of the curve α is a spacelike vector field, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\theta & 0 & \sinh\theta \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

ii) If W is a timelike vector field, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \\ \cosh\theta & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

b) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal.

i) If W is a spacelike vector field, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & 0 & -\cosh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

ii) If W is a timelike vector field, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \\ -\cosh\theta & 0 & -\sinh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length spacelike curve with spacelike binormal. Then, the natural lift $\bar{\alpha}$ of α is a timelike curve . We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$.

Let the natural lift $\bar{\alpha}$ of α is a timelike curve.

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

Let $\alpha(s)$, $L_1 \leq s \leq L_2$, be an arc length spacelike curve with timelike binormal curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike or spacelike binormal. We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$.

a) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike binormal.

i) If the Darboux vector W of the curve α is a timelike vector field, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

ii) If W is a spacelike vector field, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sinh\theta & 0 & \cosh\theta \\ -\cosh\theta & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

b) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelike binormal.

i) If W is a timelike vector field, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ \sinh\theta & 0 & -\cosh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

ii) If W is a spacelike vector field, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sinh\theta & 0 & \cosh\theta \\ \cosh\theta & 0 & -\sinh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

The linear operator $S : T_P(M) \longrightarrow T_P(M)$ defined by $S(X) = -\bar{D}_X N$ at each point $P \in M$ is called the shape operator derived from N. A regular curve α on M is said to be a line of curvature of M if for all $p \in \alpha$ the tangent line of α is a principal direction at p. According to this definition, the differential equation of the line of curvature on M is $S(T) = \lambda T$, $\lambda \neq 0$, where S is the shape operator of M.

A parametric curve $\alpha(s)$ is a curve on a surface $P = P(s,t)$ in $\mathbb{R}^3_1$ that has a constant s or t parameter value, that is, there exists a parameter s_0 or t_0 such that

$$\alpha(s) = P(s, t_0) \ \ or \ \ \alpha(t) = P(s_0, t).$$

2. Surface family with a common natural line of curvature lift of a spacelike curve with spacelike binormal in Minkowski 3-space

Suppose we are given a 3-dimensional spacelike curve with spacelike binormal $\alpha(s)$, $L_1 \leq s \leq L_2$, in which s is the arc length and $\|\alpha''(s)\| \neq 0$, $L_1 \leq s \leq L_2$. Let $\bar{\alpha}(s)$, $L_1 \leq s \leq L_2$, be the natural lift of the given curve $\alpha(s)$. Now, $\bar{\alpha}$ is a timelike curve.

Surface family that interpolates $\bar{\alpha}(s)$ as a common curve is given in the parametric form as

$$P(s,t) = \bar{\alpha}(s) + u(s,t)\,\bar{T}(s) + v(s,t)\,\bar{N}(s) + w(s,t)\,\bar{B}(s),$$

where $u(s,t)$, $v(s,t)$ and $w(s,t)$ are C^1 functions, called *marching-scale functions,* and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ is the Frenet frame of the curve $\bar{\alpha}$.

Remark 2.1. *Observe that choosing different marching-scale functions yields different surfaces possessing $\bar{\alpha}(s)$ as a common curve. Our goal is to find the necessary and sufficient conditions for which the curve $\bar{\alpha}(s)$ is line of curvature on the surface $P(s,t)$.*

Firstly, as $\bar{\alpha}(s)$ is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0,\ L_1 \le s \le L_2,\ T_1 \le t_0 \le T_2.$$

Theorem 2.1. *A necessary and sufficient condition that a surface curve be a line of curvature is that the surface normals along the curve from a developable surface.*

Let $n_1(s) = \cos\varphi \bar{N}(s) + \sin\varphi \bar{B}(s)$ be a vector orthogonal to the curve $\bar{\alpha}(s)$, where $\varphi = \varphi(s)$ is the Lorentzian spacelike angle between $\bar{N}$ and n_1.

$$n_1(s) = \cos\varphi \bar{N}(s) + \sin\varphi \bar{B}(s) = \cos(\varphi - \theta)\,T(s) + \sin(\varphi - \theta)\,B(s).$$

The curve $\bar{\alpha}(s)$ is a line of curvature on the surface $P(s,t)$ if only if n_1 is parallel to the normal vector $n(s,t)$ of the surface $P(s,t)$ and the ruled surface

$$R(s,t) = \bar{\alpha}(s) + t n_1(s),\ L_1 \le s \le L_2$$

is developable.

The normal vector field of $P(s,t)$ can be written as

$$n(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}.$$

Along the curve $\bar{\alpha}$ the normal vector field reduces to,

$$n(s,t_0) = \kappa \left[\frac{\partial w}{\partial t}(s,t_0)\,\bar{N}(s) - \frac{\partial v}{\partial t}(s,t_0)\,\bar{B}(s) \right],$$

where κ is the curvature of the curve α. This follows that $n_1(s) \parallel n(s,t)$, $L_1 \le s \le L_2$, if and only if there exits a function $\mu(s) \neq 0$ such that

$$\frac{\partial w}{\partial t}(s,t_0) = \mu(s)\cos\varphi(s),\ \frac{\partial v}{\partial t}(s,t_0) = -\mu(s)\sin\varphi(s).$$

Secondly, $P(s,t)$ is developable if and only if $\det(\bar{\alpha}', n_1, n_1') = 0$. After simple computation, we have

$$\det\left(\bar{\alpha}', n_1, n_1'\right) = 0 \iff \varphi - \theta = \text{constant}.$$

Theorem 2.2. *The natural lift curve $\bar{\alpha}(s)$ is a line of curvature on the surface $P(s,t)$ if the followings are satisfied:*

$$\begin{cases} u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0, \\ \varphi(s) - \theta(s) = constant, \\ \frac{\partial w}{\partial t}(s,t_0) = \mu(s)\cos\varphi(s), \frac{\partial v}{\partial t}(s,t_0) = -\mu(s)\sin\varphi(s), \end{cases}$$

where $L_1 \le s \le L_2, \exists t_0 \in [T_1, T_2]$, $\mu(s) \neq 0$.

Example 2.1. *Let $\alpha(s) = \frac{1}{2}\left(\cosh\left(\sqrt{2}s\right), \sqrt{2}s, \sinh\left(\sqrt{2}s\right)\right)$ be a spacelike curve with spacelike binormal. It is easy to show that*

$$\begin{aligned} T(s) &= \frac{\sqrt{2}}{2}\left(\sinh\left(\sqrt{2}s\right), 1, \cosh\left(\sqrt{2}s\right)\right), \\ N(s) &= \left(\cosh\left(\sqrt{2}s\right), 0, \sinh\left(\sqrt{2}s\right)\right), \\ B(s) &= \frac{\sqrt{2}}{2}\left(-\sinh\left(\sqrt{2}s\right), 1, -\cosh\left(\sqrt{2}s\right)\right). \end{aligned}$$

The natural lift $\bar{\alpha}(s) = \frac{\sqrt{2}}{2}\left(\sinh\left(\sqrt{2}s\right), 1, \cosh\left(\sqrt{2}s\right)\right)$ of α is a spacelike curve with timelike binormal and its Frenet vectors

$$\begin{aligned} \bar{T}(s) &= \left(\cosh\left(\sqrt{2}s\right), 0, \sinh\left(\sqrt{2}s\right)\right) \\ \bar{N}(s) &= \left(\sinh\left(\sqrt{2}s\right), 0, \cosh\left(\sqrt{2}s\right)\right) \\ \bar{B}(s) &= (0, -1, 0) \end{aligned}$$

If we let marching scale functions as $u(s,t) = \frac{\sqrt{2}}{2}t^2$, $v(s,t) = \frac{\sqrt{2}}{2}t\cos\left(\sqrt{2}t\right)$, $w(s,t) = \frac{\sqrt{2}}{2}t\sin\left(\sqrt{2}t\right)$, we get the surface

$$\begin{aligned} P(s,t) &= \bar{\alpha}(s) + u(s,t)\bar{T}(s) + v(s,t)\bar{N}(s) + w(s,t)\bar{B}(s) \\ &= \frac{\sqrt{2}}{2}\Big(\left(1 + t\cos\left(\sqrt{2}t\right)\right)\sinh\left(\sqrt{2}s\right) + t^2\cosh\left(\sqrt{2}s\right), \\ &\quad 1 - t\sin\left(\sqrt{2}t\right), t^2\sinh\left(\sqrt{2}s\right) + \left(1 + t\cos\left(\sqrt{2}t\right)\right)\cosh\left(\sqrt{2}s\right)\Big), \end{aligned}$$

$-1 \le s \le 1$, $-1 \le t \le 1$ *as a member of the surface family possessing the natural lift of the given spacelike curve as a line of curvature (Figure 1).*

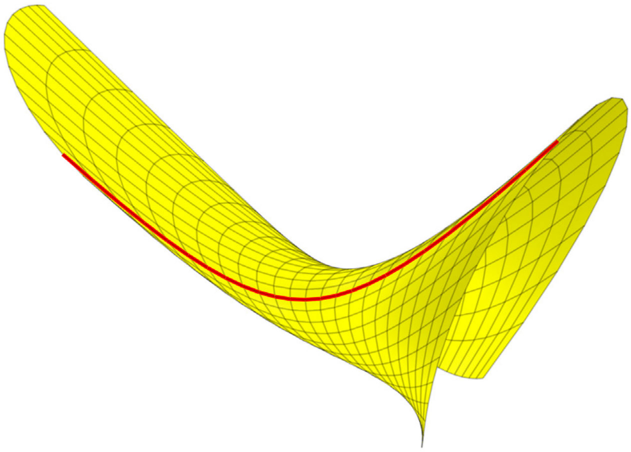

Figure 1. The surface P(s,t) possessing the natural lift (red in color) of the given curve as a line of curvature.

3. Concluding remarks

In this article, sufficient condition for a surface family possessing the natural lift of a given spacelike curve with spacelike binormal as a line of curvature is given.

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